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The chiral phase transition at high baryon density from nonperturbative flow equations*

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Abstract. We investigate the chiral phase transition at high baryon number density within the linear quark meson model for two flavors. The method we employ is based on an exact renormalization group equation for the free energy. Truncated nonperturbative flow equations are derived at nonzero chemical potential and temperature. Whereas the renormalization group flow leads to spontaneous chiral symmetry breaking in vacuum, we find a chiral symmetry restoring first order transition at high density. Combined with previous investigations at nonzero temperature, the result implies the presence of a tricritical point with long–range correlations in the phase diagram.

1 Introduction

The behavior of QCD at high temperature and baryon density is of fundamental interest and has applications in cosmology, the astrophysics of neutron stars and the physics of heavy ion collisions. Over the past years, considerable progress has been achieved in our understanding of high temperature QCD, where simulations on the lattice and universality arguments played an essential role. Recently, nonperturbative flow equations, based on the Wilsonian formulation of the renormalization group, were derived for an effective linear quark meson model [1–3]. The results of this approach account for the low temperature chiral perturbation theory domain of validity as well as the high temperature domain of critical phenomena for two quark flavors [3]. The method may help to shed some light on the remaining pressing questions at high temperature, like the nature of the phase transition for realistic values of the strange quark mass.

Our knowledge of the high density properties of strongly interacting matter is rudimentary so far. There are severe problems to use standard simulation algorithms at nonzero chemical potential on the lattice because of a complex fermion determinant. Different nonperturbative methods as the Wilsonian "exact renormalization group" or Schwinger–Dyson equations seem to present promising alternatives. It is the purpose of this letter to pursue the former approach based on an exact nonperturbative flow equation for a scale dependent effective action Γ_k [1], which is the generating functional of the $1PI$ Green functions in the presence of an infrared momentum cutoff $\sim k$. The renormalization group flow for the average action Γ_k interpolates between a given short distance or classical action S and the standard effective action Γ, which is obtained by following the flow for Γ_k to $k = 0$, thus removing the infrared cutoff in the end

QCD at nonzero baryon density is expected to have a rich phase structure with different possible phase transitions as the density is increased. A prominent example is the liquid–gas nuclear transition which has been studied in low energy heavy ion collisions. Large efforts in ultra–relativistic heavy ion collision experiments focus on a transition at high temperatures and/or densities into what is generally known as the quark–gluon plasma – a new phase of matter in which color is screened rather than confined and chiral symmetry is restored. In addition to the nuclear and quark matter phases a number of possibilities like the formation of meson condensates or strange quark matter have been investigated. Recently, an extensive discussion has focused on the symmetry of the high density ground state, where condensates of quark Cooper pairs break the color symmetry of QCD spontaneously and lead to a superconducting phase [4,5].

We will concentrate in this work on the fundamental question about the nature of the high density transition that restores chiral symmetry. Our aim is a renormalization group study of the chiral phase transition in a simple model that allows rough quantitative estimates and which has been studied in the past by alternative methods. We consider here an effective linear quark meson model. This quark meson model can be rewritten, using

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a Hubbard-Stratonovitch transformation, as a fermionic model in which the fermions interact via a momentum dependent four-fermion interaction. The model can, therefore, be viewed as a generalization of the Nambu–Jona-Lasinio model for QCD [7] which employs a point-like four-fermion interaction. Calculations at nonzero baryon density are often based on a mean field approximation. The method we employ in this work allows us to go beyond mean field in a feasible way. We compute the effective potential U as a function of the order parameter $\sim \langle\overline{\psi}\psi\rangle$ for chiral symmetry breaking and the chemical potential μ for quark number. Our approximation involves the most general order parameter dependence of the potential consistent with the symmetries.

Recently, the order of the high density chiral transition has raised considerable interest. One can argue on general grounds [5,8] that if two–flavor QCD has a second order transition (crossover) at high temperatures and a first order transition at high densities, then there exists a tricritical point (critical endpoint) with long–range correlations in the phase diagram. The physics around this point is governed by universality and may allow for distinctive signatures in heavy ion collisions [5,8,9]. It was claimed [7] that within NJL models the order of the chiral phase transition is ambiguous. We show that in the renormalization group treatment beyond the mean field approximation the order of the chiral phase transition can be fixed within the models under investigation. For this purpose, a crucial observation is the strong attraction of the flow to partial infrared fixed points [2,3]. The two remaining relevant or marginal parameters can be fixed by the phenomenological values of f_π and the constituent quark mass. For two massless quark flavors we find that chiral symmetry restoration occurs via a first order transition between a phase with low baryon density and a high density phase.

We emphasize, nevertheless, that the linear quark meson model captures the low density properties of QCD only incompletely since the effects of confinement are not included. In particular, for a discussion of the liquid–gas nuclear transition the inclusion of nucleon degrees of freedom seems mandatory [10]. The properties of high density nuclear matter could also modify the high density chiral phase transition in QCD as compared to the linear quark meson model [10]. Despite these limitations a first renormalization group investigation of the linear quark meson model seems worthwhile, both, for exploring the potential of the method for nonzero chemical potential as well as for the analogies of the model with QCD.

2 Linear quark meson model

As an effective model for the chiral properties of QCD we consider for scales k below a "compositeness scale" k_Φ a description in terms of quark and (pseudo–)scalar meson degrees of freedom [11,2,3]. All other degrees of freedom are assumed to be integrated out. The scalar and pseudoscalar mesons are described by a complex field Φ which transforms as $(\overline{\mathbf{N}},\mathbf{N})$ under the chiral flavor group $SU_L(N)\times SU_R(N)$ with N the number of light quark flavors. Chiral symmetry breaking to a vector–like subgroup $SU_V(N)$ occurs through a nonvanishing chiral condensate $\langle\Phi^{ab}\rangle = \overline{\sigma}_0\delta^{ab}$.

We will restrict our discussion here to two massless quarks. A more realistic treatment would have to include the small up and down quark masses and the finite, but much heavier, strange quark mass. In the vacuum the scalar iso–triplet as well as the pseudoscalar singlet (associated with the η' meson) contained in Φ are significantly heavier than the pions and the sigma meson. This property even holds in the symmetric phase where the pions are no longer Goldstone bosons. We take this as a guide also at nonzero density and decouple these degrees of freedom. For $N=2$ this can be achieved in a chirally invariant way. This results in a parameterization of Φ in terms of the σ–meson field and the three pion field components π^l as

$$\Phi = \frac{1}{2}\left(\sigma + i\pi^l\tau_l\right) \tag{1}$$

where τ_l $(l=1,2,3)$ denote the Pauli matrices.

Closed nonperturbative flow equations follow from the exact renormalization group equation by a suitable truncation of the effective average action. We consider here a rather simple truncation of the momentum dependence in terms of standard kinetic terms parameterized by a running meson wave function renormalization constant $Z_{\Phi,k}$ and a scale dependent Yukawa coupling $\overline{h}_k$. For the meson interaction we consider the most general form of the scalar potential U_k consistent with the symmetries. For two flavors chiral symmetry implies that U_k depends on only one invariant $\overline{\rho} = \mathrm{Tr}(\Phi^\dagger\Phi)$. The effective potential $U = \lim_{k\to 0} U_k = \lim_{k\to 0}\Gamma_k T/V$ is the relevant quantity for a study of the spontaneous breaking of chiral symmetry and is obtained for constant field configurations. At its minimum $\overline{\rho}_0$ the effective potential is related to the energy density ϵ, the entropy density s, the quark number density n_q and the pressure p by

$$U(\overline{\rho}_0;\mu,T) = \epsilon - Ts - \mu n_q = -p\,. \tag{2}$$

Our truncation omits any nontrivial momentum dependence of the effective quark and meson propagators or the interactions. In particular, it clearly does not account for confinement effects. At nonzero temperature T and chemical potential μ associated to the conserved quark number our Ansatz for Γ_k reads

$$\begin{aligned}\Gamma_k = \int_0^{1/T} dx^0 \int d^3x \Bigg\{ & i\overline{\psi}^a(\gamma^\mu\partial_\mu + \mu\gamma^0)\psi_a + \overline{h}_k\overline{\psi}^a \\ & \times\left[\frac{1+\gamma^5}{2}{\Phi_a}^b - \frac{1-\gamma^5}{2}{(\Phi^\dagger)_a}^b\right]\psi_b \\ & + Z_{\Phi,k}\partial_\mu\Phi^*_{ab}\partial^\mu\Phi^{ab} + U_k(\overline{\rho};\mu,T)\Bigg\}\,. \end{aligned} \tag{3}$$

We note that in the Euclidean formalism nonzero temperature results in (anti–)periodic boundary conditions for (fermionic) bosonic fields in the Euclidean time direction with periodicity $1/T$. A nonzero chemical potential μ to lowest order results in the term $\sim i\mu\overline{\psi}^a\gamma^0\psi_a$ appearing on the right hand side of (3). Our approximation neglects the dependence of $Z_{\Phi,k}$ and $\overline{h}_k$ on μ and T. We also neglect a possible difference in normalization of the quark kinetic term and the baryon number current. The form of the effective action at the compositeness scale, $\Gamma_{k_\Phi}[\psi,\Phi]$, serves as an initial value for the renormalization group flow of $\Gamma_k[\psi,\Phi]$ for $k < k_\Phi$. We will consider here the case that $Z_{\Phi,k_\Phi} \ll 1$. The limiting case $Z_{\Phi,k_\Phi} = 0$ can be considered as a solution of the corresponding Nambu–Jona-Lasinio model where the effective point-like four–fermion interaction has been eliminated by the introduction of auxiliary meson fields.

There is a substantial caveat concerning the approximation (3) at nonzero density. At sufficiently high density diquark condensates form, opening up a gap at the quark Fermi surfaces [4,5]. In order to describe this phenomenon in the present framework, the above Ansatz for Γ_k has to be extended to include diquark degrees of freedom. However, a nonzero diquark condensate only marginally affects the equation of state [4,5]. In particular, it has been shown in [5] in a mean field approximation that the inclusion of diquark degrees of freedom hardly changes the behavior of the chiral condensate at nonzero density and the order of the transition to the chirally symmetric phase. For NJL-type models diquark condensation is suppressed at low density by the presence of the chiral condensate. We therefore expect the Ansatz (3) to give a good description of the restoration of chiral symmetry within the present model. Our methods can be generalized to include diquark degrees of freedom. For simplicity we concentrate here on the chiral properties and neglect the superconducting properties at high densities. We hope that a more general treatment along the lines presented in this work will yield valuable information beyond a mean field approximation for the superconducting phase as well.

3 Flow equation for the effective potential

The dependence on the infrared cutoff scale k of the effective action Γ_k is given by an exact flow equation [1,12], which for fermionic fields ψ (quarks) and bosonic fields Φ (mesons) reads

$$\frac{\partial}{\partial k}\Gamma_k[\psi,\Phi] = \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial R_{kB}}{\partial k}\left(\Gamma_k^{(2)}[\psi,\Phi]+R_k\right)^{-1}\right\} - \mathrm{Tr}\left\{\frac{\partial R_{kF}}{\partial k}\left(\Gamma_k^{(2)}[\psi,\Phi]+R_k\right)^{-1}\right\}. \quad (4)$$

Here $\Gamma_k^{(2)}$ is the matrix of second functional derivatives of Γ_k with respect to both fermionic and bosonic field components. The first trace on the right hand side of (4) effectively runs only over the bosonic degrees of freedom. It implies a momentum integration and a summation over flavor indices. The second trace runs over the fermionic degrees of freedom and contains in addition a summation over Dirac and color indices. The exact flow equation closely resembles a one–loop equation with the difference that the full inverse propagator $\Gamma_k^{(2)}$ at the scale k appears instead of the classical propagator. The infrared cutoff function R_k has a block substructure with entries R_{kB} and R_{kF}. Here R_{kB} denotes the infrared cutoff function for the bosonic fields and we employ an exponential cutoff function

$$R_{kB} = \frac{Z_{\Phi,k}q^2}{\exp(q^2/k^2)-1}\,. \quad (5)$$

With this choice, R_{kB} acts as an additional mass term $R_{kB} \simeq Z_{\Phi,k}k^2$ for the low momentum $q^2 \ll k^2$ modes. The infrared cutoff function for the fermions R_{kF} should be consistent with chiral symmetries. This can be achieved if R_{kF} has the same Lorentz structure as the kinetic term for free fermions [12]. In presence of a chemical potential μ we use

$$R_{kF} = (\gamma^\mu q_\mu + i\mu\gamma^0)r_{kF}\,. \quad (6)$$

The effective squared inverse fermionic propagator is then of the form

$$\begin{aligned} P_{kF} &= [(q_0+i\mu)^2+\vec{q}^{\,2}](1+r_{kF})^2 \\ &= (q_0+i\mu)^2+\vec{q}^{\,2}+k^2\Theta(k_\Phi^2-(q_0+i\mu)^2-\vec{q}^{\,2}), \end{aligned} \quad (7)$$

where the second line defines r_{kF} and one observes that the fermionic infrared cutoff acts as an additional mass–like term[1] $\sim k^2$ Here the Θ–function can be thought of as the limit of some suitably regularized function, e.g. $\Theta^\epsilon = [\exp\{(q_0+i\mu)^2+\vec{q}^{\,2}-k_\Phi^2\}/\epsilon+1]^{-1}$.

We compute the flow equation for the effective potential U_k from equation (4) using the Ansatz (3) for Γ_k and we introduce a renormalized field $\rho = Z_{\Phi,k}\overline{\rho}$ and Yukawa coupling $h_k = Z_{\Phi,k}^{-1/2}\overline{h}_k$. The flow equation for U_k

$$\frac{\partial}{\partial k}U_k(\rho;T,\mu) = \frac{\partial}{\partial k}U_{kB}(\rho;T,\mu) + \frac{\partial}{\partial k}U_{kF}(\rho;T,\mu) \quad (8)$$

obtains contributions from bosonic and fermionic fluctuations, respectively,

$$\begin{aligned} \frac{\partial}{\partial k}U_{kB}(\rho;T,\mu) &= \frac{1}{2}T\sum_n \int_{-\infty}^{\infty}\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{Z_{\Phi,k}} \\ &\times \frac{\partial R_{kB}(q^2)}{\partial k}\left\{\frac{3}{P_{kB}\,(q^2)+U_k'(\rho;T,\mu)}\right. \\ &\left. + \frac{1}{P_{kB}\,(q^2)+U_k'(\rho;T,\mu)+2\rho U_k''(\rho;T,\mu)}\right\}, \end{aligned} \quad (9)$$

[1] The exponential form (5) of the cutoff function R_{kB} renders the first term on the right hand side of (4) both infrared and ultraviolet finite. No need for an additional ultraviolet regularization arises in this case. This is replaced by the necessary specification of an initial value Γ_{k_Φ} at the scale k_Φ. Here k_Φ is associated with a physical ultraviolet cutoff in the sense that effectively all fluctuations with $q^2 > k_\Phi^2$ are already included in Γ_{k_Φ}. A similar property for the fermionic contribution is achieved by the Θ–function in (7).

$$\frac{\partial}{\partial k}U_{kF}(\rho;T,\mu) = -8N_c T\sum_n \int_{-\infty}^{\infty} \frac{d^3\vec{q}}{(2\pi)^3} \times \frac{k\,\Theta(k_\Phi^2-(q_0+i\mu)^2-\vec{q}^2)}{P_{kF}\left((q_0+i\mu)^2+\vec{q}^2\right)+h_k^2\rho/2}\,. \quad (10)$$

Here $q^2=q_0^2+\vec{q}^2$ with $q_0(n)=2n\pi T$ for bosons, $q_0(n)=(2n+1)\pi T$ for fermions ($n\in\mathbb{Z}$) and $N_c=3$ denotes the number of colors. The scale dependent propagators on the right hand side contain the momentum dependent pieces $P_{kB}=q^2+Z_{\Phi,k}^{-1}R_k(q^2)$ and P_{kF} given by (7) as well as mass terms, where U_k', U_k'' denote the first and second derivative of the potential with respect to ρ. The only explicit dependence on the chemical potential μ appears in the fermionic contribution (10) to the flow equation for U_k. It is instructive to perform the summation of the Matsubara modes explicitly for the fermionic part. Since the flow equations only involve one momentum integration, standard techniques for one loop expressions apply and we find

$$\begin{aligned}\frac{\partial}{\partial k}U_{kF}(\rho;T,\mu) &= -8N_c\int_{-\infty}^{\infty}\frac{d^4q}{(2\pi)^4}\frac{k\,\Theta(k_\Phi^2-q^2)}{q^2+k^2+h_k^2\rho/2}\\ &+4N_c\int_{-\infty}^{\infty}\frac{d^3\vec{q}}{(2\pi)^3}\frac{k}{\sqrt{\vec{q}^2+k^2+h_k^2\rho/2}}\\ &\times\left\{\frac{1}{\exp\left[(\sqrt{\vec{q}^2+k^2+h_k^2\rho/2}-\mu)/T\right]+1}\right.\\ &\left.+\frac{1}{\exp\left[(\sqrt{\vec{q}^2+k^2+h_k^2\rho/2}+\mu)/T\right]+1}\right\} \quad (11)\end{aligned}$$

where, for simplicity, we sent $k_\Phi\to\infty$ in the μ,T dependent second integral. This is justified by the fact that in the μ,T dependent part the high momentum modes are exponentially suppressed.

For comparison, we note that within the present approach one obtains standard mean field theory results for the free energy if the meson fluctuations are neglected, $\partial U_{kB}/\partial k\equiv 0$, and the Yukawa coupling is kept constant, $h_k=h$ in (11). The remaining flow equation for the fermionic contribution could then easily be integrated with the (mean field) initial condition $U_{k_\Phi}(\rho)=\overline{m}_{k_\Phi}^2\rho$. In the following we will concentrate on the case of vanishing temperature[2]. We find (see below) that a mean field treatment yields relatively good estimates only for the μ–dependent part of the free energy $U(\rho;\mu)-U(\rho;0)$. On the other hand, mean field theory does not give a very reliable description of the vacuum properties which are important for a determination of the order of the phase transition at $\mu\neq 0$.

[2] For a solution of the flow equations at nonzero temperature and vanishing chemical potential see [3].

4 Renormalization group flow at nonzero chemical potential

In the limit of vanishing temperature one expects and observes a non–analytic behavior of the μ–dependent integrand of the fermionic contribution (11) to the flow equation for U_k because of the formation of Fermi surfaces. Indeed, the explicit μ–dependence of the flow equation reduces to a step function

$$\begin{aligned}\frac{\partial}{\partial k}U_{kF}(\rho;\mu) &= -8N_c\int_{-\infty}^{\infty}\frac{d^4q}{(2\pi)^4}\frac{k\Theta(k_\Phi^2-q^2)}{q^2+k^2+h_k^2\rho/2}\\ &+4N_c\int_{-\infty}^{\infty}\frac{d^3\vec{q}}{(2\pi)^3}\frac{k}{\sqrt{\vec{q}^2+k^2+h_k^2\rho/2}}\\ &\times\Theta\left(\mu-\sqrt{\vec{q}^2+k^2+h_k^2\rho/2}\right)\,. \quad (12)\end{aligned}$$

The quark chemical potential μ enters the bosonic part (9) of the flow equation only implicitly through the meson mass terms $U_k'(\rho;\mu)$ and $U_k'(\rho;\mu)+2\rho U_k''(\rho;\mu)$ for the pions and the σ–meson, respectively. For scales $k>\mu$ the Θ–function in (12) vanishes identically and there is no distinction between the vacuum evolution and the $\mu\neq 0$ evolution. This is due to the fact that our infrared cutoff adds to the effective quark mass $(k^2+h_k^2\rho/2)^{1/2}$. For a chemical potential smaller than this effective mass the "density" $-\partial U_k/\partial\mu$ vanishes whereas for larger μ one can view $\mu=[\vec{q}_F^{\,2}(\mu,k,\rho)+k^2+h_k^2\rho/2]^{1/2}$ as an effective Fermi energy for given k and ρ. A small infrared cutoff k removes the fluctuations with momenta in a shell close to the physical Fermi surface[3] $\mu^2-h_k^2\rho/2-k^2<q^2<\mu^2-h_{k=0}^2\rho/2$. Our flow equation realizes the general idea [13] that for $\mu\neq 0$ the lowering of the infrared cutoff $k\to 0$ should correspond to an approach to the physical Fermi surface. For a computation of the meson effective potential the approach to the Fermi surface in (12) proceeds from below and for large k the effects of the Fermi surface are absent. By lowering k one "fills the Fermi sea".

In vacuum the evolution of U_k, the Yukawa coupling h_k and the meson wave function renormalization $Z_{\Phi,k}$ have been computed earlier [2,3][4]. The flow equations for h_k and $Z_{\Phi,k}$ can be derived from (4) using the truncation (3) and one finds that for large enough k their running is well approximated by [2,3] ($k<k_\Phi$)

$$h_k^2\simeq Z_{\Phi,k}^{-1}\simeq-\frac{8\pi^2}{N_c\ln(k/k_\Phi)}\,. \quad (13)$$

[3] If one neglects the mesonic fluctuations one can perform the k–integration of the flow equation (12) in the limit of a k–independent Yukawa coupling. One recovers (for $k_\Phi^2\gg k^2+h^2\rho/2,\mu^2$) mean field theory results except for a shift in the mass, $h^2\rho/2\to h^2\rho/2+k^2$, and the fact that modes within a shell of three–momenta $\mu^2-h^2\rho/2-k^2\leq\vec{q}^{\,2}\leq\mu^2-h^2\rho/2$ are not yet included. Because of the mass shift the cutoff k also suppresses the modes with $q^2<k^2$.

[4] In [2,3] a different variant of the fermionic infrared cutoff function R_{kF} than (7) was used.

One observes a very strong Yukawa coupling h_k for k in the vicinity of k_Φ. (The Landau pole at $k = k_\Phi$ corresponds to taking $Z_{\Phi,k_\Phi} \to 0$.) This strong Yukawa coupling implies a fast approach of running couplings to partial infrared fixed points [2,3]. More precisely, for a rescaled potential $U_k(\rho)/k^4 = \sum_{n=0}^{\infty} \frac{1}{n!} u_k^{(n)} (\rho/k^2)^n$ the running dimensionless couplings $u_k^{(n)}$ for $n \geq 2$ approach values proportional to powers of the Yukawa coupling h_k, with $u_k^{(2)}/h_k^2 \simeq 1$, $u_k^{(3)}/h_k^6 \simeq -0.0164$, $u_k^{(4)}/h_k^8 \simeq 0.0105$ and similarly for the higher couplings [3]. In consequence, the details of the short distance meson interactions become unimportant and besides the marginal coupling h_k the only relevant parameter of the model corresponds to the mass parameter $u_k^{(1)}$. We note that the large value of h_k is phenomenologically suggested by the comparably large value of the constituent quark mass. The observed fixed point behavior allows us to fix the model with only two phenomenological input parameters and we use $f_\pi = 92.4\,\mathrm{MeV}$ and $300\,\mathrm{MeV} \lesssim M_q \lesssim 350\,\mathrm{MeV}$. Previous results for the evolution in vacuum [2,3] show that for scales not much smaller than $k_\Phi \simeq 650\,\mathrm{MeV}$ chiral symmetry remains unbroken. This holds down to a scale $k_{\chi SB} \simeq 450\,\mathrm{MeV}$ at which the meson potential $U_k(\rho)$ develops a minimum at $\rho_{0,k} > 0$ thus breaking chiral symmetry spontaneously. Below the chiral symmetry breaking scale running couplings are no longer governed by the partial fixed point. In particular, for $k \lesssim k_{\chi SB}$ the Yukawa coupling h_k and the meson wave function renormalization $Z_{\Phi,k}$ depend only weakly on k and approach their infrared values.

At $\mu \neq 0$ we will follow the evolution from $k = k_{\chi SB}$ to $k = 0$ and neglect the k–dependence of h_k and $Z_{\Phi,k}$ in this range. According to the above discussion the initial value $U_{k_{\chi SB}}$ is μ–independent for $\mu < k_{\chi SB}$. At this scale $U_{k_{\chi SB}}$ is well approximated by the polynomial form where the coefficients $u_{k_{\chi SB}}^{(n)}$, $n = 2, 3, 4$, are estimated by their partial infrared fixed point values and coefficients with $n > 4$ are neglected. The scalar mass parameter $u_k^{(1)}$ crosses from positive to negative values at the chiral symmetry breaking scale and thus vanishes at $k = k_{\chi SB}$. The constant $u_{k_{\chi SB}}^{(0)}$ is fixed such that the pressure of the physical vacuum is zero. We do not assume a polynomial form for $U_k(\rho;\mu)$ for $k < k_{\chi SB}$ and solve the flow equation (8) with (9), (12) numerically[5] as a partial differential equation for the potential depending on the two variables ρ and k for given μ.

In the fermionic part (12) of the flow equation the vacuum and the μ–dependent term contribute with opposite signs. This cancellation of quark fluctuations with momenta below the Fermi surface is crucial for the restoration of chiral symmetry at high density[6]. In vacuum, spontaneous chiral symmetry breaking is induced in our model by quark fluctuations which drive the scalar mass term $U_k'(\rho = 0)$ from positive to negative values at the scale $k = k_{\chi SB}$. (Meson fluctuations have the tendency to restore chiral symmetry because of the opposite relative sign, cf. (4).) As the chemical potential becomes larger than the effective mass $(k^2 + h_k^2\rho/2)^{1/2}$ quark fluctuations with momenta smaller than $\vec{q}_F^{\,2}(\mu, k, \rho) = \mu^2 - k^2 - h_k^2\rho/2$ are suppressed. Since $\vec{q}_F^{\,2}$ is monotonically decreasing with ρ for given μ and k the origin of the effective potential is particularly affected. We will see in the next section that for large enough μ this leads to a second minimum of $U_{k=0}(\rho;\mu)$ at $\rho = 0$ and a chiral symmetry restoring first order transition.

5 High density chiral phase transition

In vacuum or at zero density the effective potential $U(\sigma)$, $\sigma \equiv \sqrt{\rho/2}$, has its minimum at a nonvanishing value $\sigma_0 = f_\pi/2$ corresponding to spontaneously broken chiral symmetry. As the quark chemical potential μ increases, U can develop different local minima. The lowest minimum corresponds to the state of lowest free energy and is favored. In Fig. 1 we plot the free energy as a function of σ for different values of the chemical potential $\mu = 322.6, 324.0, 325.2\,\mathrm{MeV}$. Here the effective constituent quark mass is $M_q = 316.2\,\mathrm{MeV}$. We observe that for $\mu < M_q$ the potential at its minimum does not change with μ. Since (cf. (2))

$$n_q = -\frac{\partial U}{\partial \mu}_{|\min} \tag{14}$$

we conclude that the corresponding phase has zero density. In contrast, for a chemical potential larger than M_q we find a low density phase where chiral symmetry is still broken. The quark number density as a function of μ is shown in Fig. 2. One clearly observes the non–analytic behavior at $\mu = M_q$ which denotes the "onset" value for nonzero density. From Fig. 1 one also notices the appearance of an additional local minimum at the origin of U. As the pressure $p = -U$ increases in the low density phase with increasing μ, a critical value μ_c is reached at which there are two degenerate potential minima. Before μ can increase any further the system undergoes a first order

[5] Nonzero current quark masses, which are neglected in our approximation, result in a pion mass threshold and would effectively stop the renormalization group flow of renormalized couplings at a scale around m_π. To mimic this effect one may stop the evolution by hand at $k_f \simeq m_\pi$. We observe that our results are very insensitive to such a procedure which can be understood from the fact that a small infrared cutoff – induced by k_f or by the nonzero current quark masses – plays only a minor role for a sufficiently strong first order transition. For the results presented in the next section we use $k_f = 100\,\mathrm{MeV}$. The flow of the potential U_k around its minima and for its outer convex part stabilizes already for k somewhat larger than k_f. Fluctuations on larger length scales lead to a flattening of the barrier between the minima [15]. The approach to convexity is not relevant for the present discussion. It is treated numerically in a simplified way by enhancing the pole which appears in the momentum integral (9) for negative U_k' as k^2 approaches $-U_k'$.

[6] We note that the renormalization group investigation in [14] of a linear sigma model in $4 - \epsilon$ dimensions misses this property.

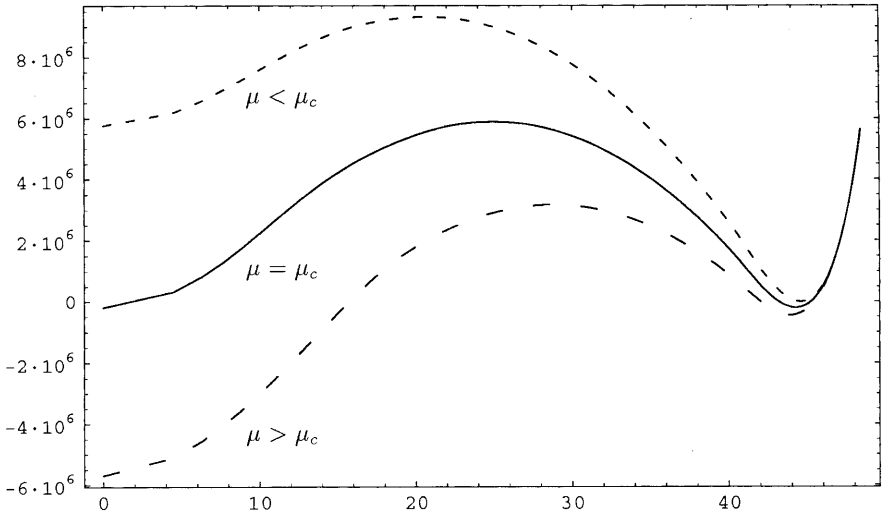

Fig. 1. The zero temperature effective potential U (in MeV^4) as a function of $\sigma \equiv (\rho/2)^{1/2}$ for different chemical potentials. One observes two degenerate minima for a critical chemical potential $\mu_c/M_q = 1.025$ corresponding to a first order phase transition at which two phases have equal pressure and can coexist ($M_q = 316.2\,\mathrm{MeV}$)

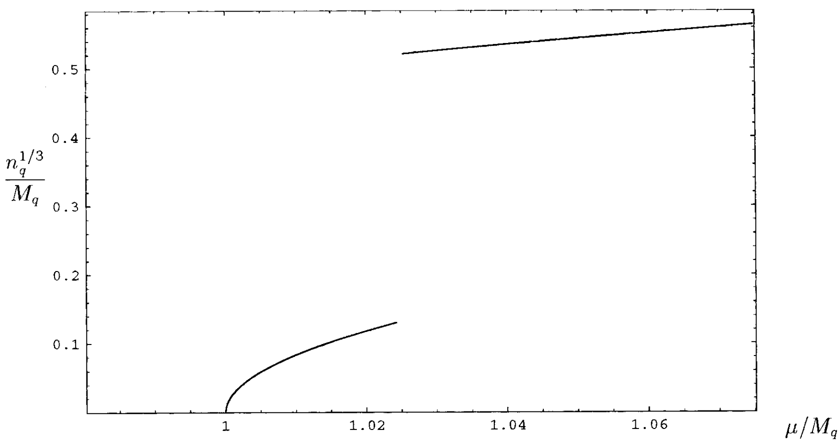

Fig. 2. The plot shows $n_q^{1/3}$, where n_q denotes the quark number density as a function of μ in units of the effective constituent quark mass ($M_q = 316.2\,\mathrm{MeV}$)

phase transition at which two phases have equal pressure and can coexist. In the high density phase chiral symmetry is restored as can be seen from the vanishing order parameter for $\mu > \mu_c$. The curvature of the potential at the origin for $\mu \gtrsim \mu_c$ is $\partial^2 U/\partial\sigma^2(\sigma = 0) \simeq (1.45 M_q)^2$. We note that the relevant scale for the first order transition is M_q. For this reason we have scaled our results for dimensionful quantities in units of M_q.

For the class of quark meson models considered here (with M_q/f_π in a realistic range around $3 - 4$) the first order nature of the high density transition has been clearly established. In particular, these models comprise the corresponding Nambu–Jona-Lasinio models where the effective fermion interaction has been eliminated by the introduction of auxiliary bosonic fields. Our treatment, which is based on a nonperturbative evolution equation for the free energy, goes beyond mean field theory and puts it into a more systematic context. For the quark meson model this method has been used in the past to study the chiral phase transition at nonzero temperature, including the critical properties near the second order transition for two massless flavors characteristic of the three dimensional $O(4)$ universality class [3]. In the chiral limit, we therefore find a second order chiral transition at zero chemical potential and a first order chiral transition at zero temperature. By continuity these transitions meet at a tricritical point in the (μ, T)–plane [5,8]. Away from the chiral limit, the second order chiral transition turns into a smooth crossover. The first order line of transitions at low temperatures now terminates in a critical endpoint with long–range correlations [5,8]. Using the method employed here a precision estimate of the universal critical equation of state of this Ising endpoint has been given in [16].

The extent to which the transition from nuclear matter to quark matter in (two-flavor) QCD differs from the transition of a quark gas to quark matter in the quark meson model or NJL–type models has to be clarified by further investigations. At the phase transition, the quark number density in the symmetric phase $n_{q,c}^{1/3} = 0.52 M_q$ turns out in this model to be not much larger than nuclear matter density, $n_{q,nuc}^{1/3} = 152\,\mathrm{MeV}$. Here the open problems are related to the description of the low density phase, where confinement effects play a crucial role, rather than the high density phase. Quite generally, the inclusion of nucleon degrees of freedom for the description of the low density phase shifts the transition to a larger chemical potential and larger baryon number density for both the nuclear and the quark matter phases [10]. The quark meson model is, therefore, not expected to give a precise description of the critical chemical potential for chiral symmetry restoration in QCD. In the present model the two-flavor chiral phase transition turns out to be first order once the free parameters are fixed by phenomenological input. Though this is a robust prediction for the model, we note that the inclusion of nucleon degrees of freedom can result in a rather weak transition or even a crossover to the chirally symmetric phase [10]. Finally, the low–density first order transition from a gas of nucleons or the vacuum to nuclear matter (nuclear gas–liquid transition) can only be understood if the low–momentum fermionic degrees of freedom are described by nucleons rather than quarks [10]. The low–density branch of Fig. 2 cannot be carried over to QCD. We emphasize that nucleon degrees of freedom can be included in the framework of nonperturbative flow equations.

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